BERNOULLI NUMBERS AND POLYNOMIALS

by

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INTRODUCTION

Bernoulli polynomials and numbers were first introduced by Jacob Bernoulli, and the Bernoulli polynomials are a special case of Appell polynomials. A set of polynomials are called Appell polynomials if they obey the relationship

$$\frac{d}{dx} A_n(x) = nA_{n-1}(x).$$

It is shown in this report that the Bernoulli polynomials obey this relationship.

Bernoulli polynomials and numbers are used in the theory of finite differences especially in the process of summation. This report will be concerned with the development of some of the more important properties of Bernoulli polynomials and numbers rather than with their usage.

THE GENERATING FUNCTION OF THE BERNOULLI POLYNOMIALS

Consider the function

$$f(x, y) = \frac{ye^{xy}}{e^y - 1} , \tag{1}$$

known as the generating function of the Bernoulli polynomials. Expand $\frac{ye^{xy}}{e^y-1}$ in a series of powers of y; its coefficients are functions of the parameter x, and they are also polynomials in x. The coefficients are defined as the Bernoulli polynomials, i.e.,

$$f(x, y) = \sum_{n=0}^{\infty} B_n(x) \frac{y^n}{n!} \tag{2}$$

where $B_n(x)$ are the Bernoulli polynomials. The generating function is defined for all y except possibly y = 0. When y = 0, the generating function is an indeterminate form, but using L'Hospital's Rule[1] it is found that $\lim_{y \to 0} f(x, y) = 1$ for any x. Hence, f(x, 0) will be defined as unity.

The first few polynomials can be determined by long division. Expanding both numerator and denominator of the generating function in Maclaurin series yields

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[1] Angus Taylor, Advanced Calculus, p. 121.

$$ye^{xy} = y + xy^2 + \frac{x^2y^3}{2!} + \frac{x^3y^4}{3!} + \cdots \tag{3}$$

and

$$e^y - 1 = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots\ . \tag{4}$$

Thus

$$f(x, y) = 1 + (x - \tfrac{1}{2})y + (\tfrac{1}{2}x^2 - \tfrac{1}{2}x + \tfrac{1}{12})y^2 + (\tfrac{x^3}{6} - \tfrac{x^2}{4} + \tfrac{x}{12})y^3 + \cdots\ . \tag{5}$$

The coefficients, $B_n(x)$, $n = 0, 1, 2, 3, \cdots$, are thus

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \tfrac{1}{2} \\
B_2(x) &= x^2 - x + \tfrac{1}{6} \\
B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{x}{2} \\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30} \\
B_5(x) &= x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{x}{6} \\
&\cdots\ .
\end{aligned} \tag{6}$$

Using long division to find the Bernoulli polynomials becomes cumbersome very rapidly, and a simpler method for determining them will be developed later.

The expansion of the generating function as a power series in y raises the question of convergence. The generating function may be rewritten as

$$f(x, y) = \frac{e^{xy}}{1 + \frac{y}{2!} + \frac{y^2}{3!} + \cdots} \quad . \tag{7}$$

Letting

$$Y = \frac{y}{2!} + \frac{y^2}{3!} + \frac{y^3}{4!} + \cdots \quad , \tag{8}$$

equation (7) becomes

$$f(x, y) = \frac{e^{xy}}{1 + Y} \quad . \tag{9}$$

Expansion of $(1 + Y)^{-1}$ by means of the Binomial Theorem yields

$$f(x, y) = e^{xy}(1 - Y + Y^2 - Y^3 + \cdots). \tag{10}$$

The generating function is now expressed as the product of two convergent series, because e^{xy} converges for all xy and $(1 + Y)^{-1}$ converges for all $|Y| < 1$. Since the product of two convergent series converges within their common interval of convergence, a positive number r can be found such that the power series expansion of the generating function converges for $|y| < r$.

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Setting x = 0 in equation (2) gives

$$f(0, y) = \frac{y}{e^y - 1} = \sum_{n=0}^{\infty} B_n(0) \frac{y^n}{n!} \quad . \tag{11}$$

The numbers $B_n(0)$ are called the Bernoulli numbers and are denoted by B_n, that is, $B_n(0) = B_n$; $n = 0, 1, 2, 3, \cdots$. From equations (6) it is seen that

$$B_0(0) = 1$$

$$B_1(0) = -\frac{1}{2}$$

$$B_2(0) = \frac{1}{6}$$

$$B_3(0) = 0$$

$$B_4(0) = -\frac{1}{30}$$

$$B_5(0) = 0$$

$$\cdots .$$

A number of relations concerning Bernoulli numbers and polynomials follows.

<u>Theorem 1:</u> $B_{2n+1} = 0$, $n = 1, 2, 3, \cdots$.

<u>Proof:</u> Referring to equation (11),

$$f(0, y) = \frac{y}{e^y - 1}$$

$$f(0, -y) = \frac{-y}{e^{-y} - 1}$$

$$f(0, y) - f(0, y) = -y = \sum_{n=0}^{\infty} B_n(0) \frac{y^n}{n!} - \sum_{n=0}^{\infty} B_n(0)(-1)^n \frac{y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[1 - (-1)^n\right] B_n(0) \frac{y^n}{n!} .$$

Transposing -y and partially expanding the sum, one has

$$y + 0B_0 + 2B_1 y + 0B_2 \frac{y^2}{2!} + 2B_3 \frac{y^3}{3!} + 0B_4 \frac{y^4}{4!} + \cdots = 0.$$

This may be rewritten as

$$(1 + 2B_1)y + \sum_{n=1}^{\infty} 2B_{2n+1} \frac{y^{2n+1}}{(2n+1)!} = 0.$$

Because of the linear independence of the powers of y,

$$B_1 = -\frac{1}{2}$$

$$B_{2n+1} = 0, \; n = 1, 2, 3,$$

This completes the proof.

Theorem 2: $B_1(1) = \frac{1}{2}$, $B_n(1) = B_n(0)$; $n = 0, 2, 3, 4, \cdots$.

Proof: Set $x = 1$ in equation (1). Now

$$f(1, y) = \frac{ye^y}{e^y - 1} = \sum_{n=0}^{\infty} B_n(1) \frac{y^n}{n!} . \tag{12}$$

Also

$$\frac{ye^y}{e^y - 1} = y + \frac{y}{e^y - 1} = y + \sum_{n=0}^{\infty} B_n(0) \frac{y^n}{n!} . \tag{13}$$

Equating the right-hand members of equations (12) and (13), one has

$$y + \sum_{n=0}^{\infty} B_n(0) \frac{y^n}{n!} = \sum_{n=0}^{\infty} B_n(1) \frac{y^n}{n!}$$

or

$$y + \sum_{n=0}^{\infty} \left[B_n(0) - B_n(1)\right] \frac{y^n}{n!} = 0.$$

Partially expanding the sum, one has

$$B_0(0) - B_0(1) \quad + \left[1 + B_1(0) - B_1(1)\right] y + \sum_{n=2}^{\infty} \left[B_n(0) - B_n(1)\right] \frac{y^n}{n!} = 0.$$

Because of the linear independence of powers of y,

$$B_0(1) = B_0(0)$$

$$B_1(1) = 1 + B_1(0) = \frac{1}{2}$$

$$B_n(1) = B_n(0), \; n = 2, 3, 4, \cdots .$$

This completes the proof.

<u>Theorem 3:</u> $B_p(x) = \sum_{n=0}^{p} C(p, n)\, B_n(0)\, x^{p-n}$, where C(p, n) is the symbol denoting the number of combinations of p elements taken n at a time.

<u>Proof:</u> Recall the generating function of the Bernoulli polynomials and write it as

$$\sum_{p=0}^{\infty} B_p(x) \frac{y^p}{p!} = \frac{y}{e^y - 1} e^{xy}$$

$$= \sum_{n=0}^{\infty} \frac{B_n y^n}{n!} \sum_{m=0}^{\infty} \frac{x^m y^m}{m!} \quad .$$

Noting that the Maclaurin series expansion of e^{xy} converges absolutely, and the Maclaurin series expansion of $\frac{y}{e^y - 1}$ converges, a Cauchy product[2] may be formed. Forming the Cauchy product yields

$$\sum_{n=0}^{\infty} \frac{B_n y^n}{n!} \sum_{m=0}^{\infty} \frac{x^m y^m}{m!} = \sum_{p=0}^{\infty} \sum_{n=0}^{p} \frac{B_n x^{p-n} y^p}{n!\,(p-n)!} .$$

Now

$$\sum_{p=0}^{\infty} B_p(x) \frac{y^p}{p!} = \sum_{p=0}^{\infty} \sum_{n=0}^{p} \frac{B_n x^{p-n} y^p}{n!\,(p-n)!} ,$$

and equating corresponding coefficients of y gives the relationship,

$$\frac{B_p(x)}{p!} = \sum_{n=0}^{p} \frac{B_n x^{p-n}}{n!\,(p-n)!} .$$

Therefore,

$$B_p(x) = \sum_{n=0}^{p} C(p, n)\, B_n x^{p-n} .$$

By the use of Theorem 3, one can calculate the Bernoulli polynomials knowing only the Bernoulli numbers. This theorem can also be used to derive a formula which expresses the (r + 1)st Bernoulli number in terms of the first r Bernoulli numbers.

<u>Theorem 4:</u> $B_{2q} = \frac{-1}{2q+1} \sum_{n=0}^{2q-1} C(2q+1, n)\, B_n, \; q \geq 1.$

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[2] Walter Rudin, <u>Principles of Mathematical Analysis,</u> p. 58.

Proof: Consider Theorem 3, and set $p = 2q + 1$. Partially expanding the sum yields

$$B_{2q+1}(x) = \sum_{n=0}^{2q-1} C(2q+1, n)B_n x^{2q+1-n} + C(2q+1, 2q)B_{2q}x + C(2q+1, 2q+1)B_{2q+1}. \quad (14)$$

From Theorem 2,

$$B_n(1) = B_n, \quad n = 0, 2, 3, 4, \cdots .$$

From Theorems 1 and 2,

$$B_{2q+1} = B_{2q+1}(1) = 0, \quad q \geq 1.$$

Thus, for $q \geq 1$ and $x = 1$, equation (14) may be written as

$$0 = \sum_{n=0}^{2q-1} C(2q+1, n)B_n + (2q+1)B_{2q} + (1)(0).$$

Therefore,

$$B_{2q} = \frac{-1}{2q+1} \sum_{n=0}^{2q-1} C(2q+1, n)B_n .$$

Define the difference operator Δ by the relation:

$$\Delta f(x) = f(x+1) - f(x).$$

Theorem 5: $\Delta B_n(x) = nx^{n-1}$, $n = 0, 1, 2, 3, \cdots .$

Proof:

$$\Delta f(x,y) = \frac{ye^{(x+1)y}}{e^y - 1} - \frac{ye^{xy}}{e^y - 1}$$

$$= ye^{xy}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} y^n .$$

Also

$$\Delta f(x,y) = \Delta \sum_{n=0}^{\infty} B_n(x) \frac{y^n}{n!}$$

$$= \sum_{n=0}^{\infty} \Delta B_n(x) \frac{y^n}{n!}$$

$$= \sum_{n=1}^{\infty} \Delta B_n(x) \frac{y^n}{n!} ,$$

because $\Delta B_0 = 0$.

Equating the two expressions for $\Delta f(x,y)$ yields

$$\frac{\Delta B_n(x)}{n!} = \frac{x^{n-1}}{(n-1)!} .$$

Thus

$$\Delta B_n(x) = nx^{n-1}, \quad n = 0, 1, 2, 3, \cdots .$$

Define the differential operator D by the relation:

$$Df(x) = \frac{df(x)}{dx} .$$

Theorem 6: $DB_n(x) = nB_{n-1}(x), \quad n = 0, 1, 2, 3, \cdots .$

Proof: Differentiate the generating function of the Bernoulli polynomials with respect to x and obtain

$$\frac{\partial f}{\partial x}(x, y) = yf(x, y) = \sum_{t=0}^{\infty} B_t(x) \frac{y^{t+1}}{t!} \quad . \tag{15}$$

Differentiate the power series expansion of the Bernoulli polynomials with respect to x and find

$$\frac{\partial f}{\partial x}(x, y) = \sum_{n=0}^{\infty} DB_n(x) \frac{y^n}{n!} = \sum_{n=1}^{\infty} DB_n(x)\frac{y^n}{n!} \quad , \tag{16}$$

because $DB_0(x) = 0$.

Letting $n = t+1$ in equation (16) and comparing equations (15) and (16), one has

$$\sum_{t=0}^{\infty} DB_{t+1}(x) \frac{y^{t+1}}{(t+1)!} = \sum_{t=0}^{\infty} B_t(x) \frac{y^{t+1}}{t!} \quad . \tag{17}$$

Equating corresponding coefficients of y in equation (17), one finds

$$\frac{DB_{t+1}(x)}{t+1} = B_t(x),$$

which implies

$$DB_n(x) = nB_{n-1}(x).$$

Theorem 7: $\sum_{t=0}^{n-1} C(n, t)B_t(x) = nx^{n-1}$, $n = 1, 2, 3, \cdots$.

Proof: Recall that $B_k(x)$ is a polynomial of degree k. Write

$$x^n = \sum_{k=0}^{n} a_k B_k(x) = a_0 B_0(x) + \sum_{t=0}^{n-1} a_{t+1} B_{t+1}(x). \tag{18}$$

Now one must see if the a_k coefficients can be determined. Differencing both sides of equation (18), one obtains

$$\Delta x^n = (x+1)^n - x^n = \sum_{t=0}^{n-1} a_{t+1} \Delta B_{t+1}(x), \tag{19}$$

since $\Delta B_0(x) = 0$.

From Theorem 5,

$$\Delta B_{t+1}(x) = (t+1)x^t, \quad t \geq 0 \tag{20}$$

By the Binomial Theorem,

$$(x+1)^n - x^n = \sum_{t=0}^{n-1} C(n,t)x^t . \tag{21}$$

Using the results of equations (20) and (21), equation (19) may be expressed as

$$\sum_{t=0}^{n-1} C(n,t)x^t = \sum_{t=0}^{n-1} a_{t+1}(t+1)x^t.$$

Equating corresponding coefficients of x, one finds

$$a_{t+1} = \frac{C(n,t)}{t+1}, \quad t = 0, 1, 2, \cdots, n-1. \tag{22}$$

Using the results of equation (22), equation (18) becomes

$$x^n = a_0 + \sum_{t=0}^{n-1} \frac{C(n,t)}{t+1} B_{t+1}(x). \tag{23}$$

Differentiating both sides of equation (23) with respect to x, one finds

$$nx^{n-1} = \sum_{t=0}^{n-1} C(n,t)B_t(x). \tag{24}$$

This theorem enables one to express the Bernoulli polynomial of degree n in terms of Bernoulli polynomials of lower degree. This result is seen more easily by letting m = n-1 in equation (24) and partially expanding it to obtain

$$B_m(x) = x^m - \frac{1}{m+1} \sum_{t=0}^{m-1} C(m+1,t)B_t(x).$$

Theorem 8: $\zeta(2p) = \frac{(2\pi)^{2p}(-1)^{p-1}}{2(2p)!} B_{2p},\ p = 1, 2, 3, \cdots .$

$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is the Riemann-Zeta function.

Proof:

$$\begin{aligned} \coth y &= \frac{e^y + e^{-y}}{e^y - e^{-y}} \\ &= \frac{e^y}{e^y - e^{-y}} + \frac{e^{-y}}{e^y - e^{-y}} \\ &= \frac{1}{e^{2y} - 1} + \frac{1}{1 - e^{-2y}} . \end{aligned} \tag{25}$$

Multiply both sides of equation (25) by 2y, and one has

$$2y \coth y = \frac{2y}{e^{2y} - 1} + \frac{2y}{1 - e^{-2y}} .$$

Recall the generating function for the Bernoulli numbers, and 2y coth y may be expressed as

$$2y \coth y = \sum_{n=0}^{\infty} B_n \frac{(2y)^n}{n!} + \sum_{n=0}^{\infty} B_n \frac{(-2y)^n}{n!} .$$

For odd powers of n the sums cancel and thus

$$y \coth y = \sum_{p=0}^{\infty} B_{2p} \frac{(2y)^{2p}}{(2p)!} . \tag{26}$$

Using the relationship[3]

$$\pi \cot \pi u - \frac{1}{u} = -2 \sum_{n=1}^{\infty} \frac{u}{n^2 - u^2} , \quad |u| < 1,$$

and cot iu = -i coth u, let u = ix and

$$\pi \coth \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{n^2 + x^2} . \tag{27}$$

Let $x = \frac{t}{2\pi}$ and equation (27) becomes

$$\pi \coth \frac{t}{2} = \frac{2\pi}{t} + \sum_{n=1}^{\infty} \frac{\frac{t}{\pi}}{n^2 + \frac{t^2}{(2\pi)^2}} . \tag{28}$$

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[3]Kenneth S. Miller, An Introduction to the Calculus of Finite Differences and Difference Equations, p. 54.

Multiply both sides of equation (28) by $\frac{t}{2\pi}$, and it becomes

$$\frac{t}{2}\coth\frac{t}{2} = 1 + 2\sum_{n=1}^{\infty}\frac{\left(\frac{t}{2\pi}\right)^2}{n^2+\left(\frac{t}{2\pi}\right)^2} \tag{29}$$

Let y = t/2 in equation (26). Then

$$\frac{t}{2}\coth\frac{t}{2} = \sum_{p=0}^{\infty} B_{2p}\frac{t^{2p}}{(2p)!} \;. \tag{30}$$

Equating the right-hand members of equation (29) and (30), one finds that

$$\sum_{p=0}^{\infty} B_{2p}\frac{t^{2p}}{(2p)!} = 1 + 2\sum_{n=1}^{\infty}\frac{\left(\frac{t}{2\pi}\right)^2}{n^2+\left(\frac{t}{2\pi}\right)^2} \;. \tag{31}$$

Divide both numerator and denominator of $\dfrac{\left(\frac{t}{2\pi}\right)^2}{n^2+\left(\frac{t}{2\pi}\right)^2}$ by n^2, and obtain

$$\frac{\left(\frac{t}{2\pi}\right)^2}{n^2+\left(\frac{t}{2\pi}\right)^2} = \frac{\left(\frac{t}{2n\pi}\right)^2}{1+\left(\frac{t}{2n\pi}\right)^2} = \sum_{p=1}^{\infty}(-1)^{p-1}\left(\frac{t}{2n\pi}\right)^{2p} \;.$$

The expression $\dfrac{\left(\frac{t}{2n\pi}\right)^2}{1+\left(\frac{t}{2n\pi}\right)^2}$ represents the sum of a geometric progression

with common ratio $-\left(\frac{t}{2n\pi}\right)^2$; therefore, it can be replaced by

$$\sum_{p=1}^{\infty} (-1)^{p-1} \left(\frac{t}{2n\pi}\right)^{2p} .$$

Using this result, equation (31) becomes

$$1 + \sum_{p=1}^{\infty} B_{2p} \frac{t^{2p}}{(2p)!} = 1 + 2 \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} (-1)^{p-1} \left(\frac{t}{2n\pi}\right)^{2p},$$

and, on equating corresponding coefficients of t,

$$\frac{B_{2p}}{(2p)!} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{p-1}}{(2n\pi)^{2p}}, \quad p > 0.$$

Dividing through by 2,

$$\frac{B_{2p}}{2(2p)!} = \frac{(-1)^{p-1}}{(2\pi)^{2p}} \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{(-1)^{p-1}}{(2\pi)^{2p}} \zeta(2p) .$$

Therefore

$$\zeta(2p) = \frac{(2\pi)^{2p}(-1)^{p-1}B_{2p}}{2(2p)!} .$$

There is no known closed form for $\zeta(p)$ when p is an odd integer, but using Theorem 8, one can sum $\zeta(p)$ when p is an even integer.

<u>Theorem 9</u>: $B_n(1 - x) = (-1)^n B_n(x)$. This theorem is known as the complementary theorem.

Proof: Refer to the generating function of the Bernoulli polynomials and replace y by -y. Thus

$$f(x, -y) = \frac{-ye^{-xy}}{e^{-y}-1} = \sum_{n=0}^{\infty} B_n(x)(-1)^n \frac{y^n}{n!} . \tag{32}$$

Multiply numerator and denominator of $\frac{-ye^{-xy}}{e^{-y}-1}$ by e^y. Then

$$f(x, -y) = \frac{-ye^{y(1-x)}}{e^y-1} = \frac{ye^{y(1-x)}}{1-e^y} = \sum_{n=0}^{\infty} B_n(1-x) \frac{y^n}{n!} . \tag{33}$$

Equating the right-hand members of equations (32) and (33) yields

$$B_n(1-x) = (-1)^n B_n(x) .$$

Corollary: $B_n(1/2) = 0$ for n odd.

The proof is immediate, on setting $x = 1/2$ in the formula of Theorem 9.

Theorem 10: There is no zero of $B_{2n+1}(x)$ within the open interval $(0, 1)$ other than $1/2$.

Proof: For $n = 0$, $B_1(x) = x - 1/2$, and the theorem is true. From the corollary to Theorem 9, $B_{2n+1}(1/2) = 0$, and from Theorems 1 and 2

$$B_{2n+1}(0) = B_{2n+1}(1) = 0, \; n = 1, 2, 3, \cdots .$$

Suppose that $B_{2(n+1)+1}(p) = 0$, where p lies in the open interval $(0, 1)$

and is not equal to 1/2. Then $B_{2(n+1)+1}(x)$ has four zeros in the closed interval $[0, 1]$, and by Rolle's Theorem [3] $DB_{2(n+1)+1}(x)$ has at least three zeros in the open interval (0, 1). By Rolle's Theorem again $D^2B_{2(n+1)+1}(x)$ has at least two zeros in the open interval (0, 1). From Theorem 6

$$D^2B_{2(n+1)+1}(x) = (2n+3)\, DB_{2(n+1)}(x) = (2n+3)(2n+1)\, B_{2n+1}(x),$$

and $B_{2n+1}(x)$ vanishes at $x = 0$, $1/2$ and 1 as previously stated. By assumption, $B_{2n+1}(x)$ has four zeros in the closed interval $[0, 1]$. Consider $B_{2n+1}(x)$ when $n = 1$. The representation of this is

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2} \;,$$

and its zeros are $x = 0$, $x = 1/2$, and $x = 1$. Thus $B_3(x)$ has only one zero in the open interval (0, 1). Since $B_3(x)$ can be expressed in terms of derivatives of higher order Bernoulli polynomials of odd degree, it is impossible for these higher order polynomials to have two zeros in the open interval (0, 1) when $B_3(x)$ has only one zero in this interval. This furnishes a contradiction to the assumption that $B_{2(n+1)+1}(p) = 0$ with $0 < p < 1$ and p not equal to 1/2.

<u>Theorem 11:</u> $B_{2n}(x) - B_{2n}$ retains the same sign over the open interval (0, 1).

Proof: Let

$$f(x) = B_{2n}(x) - B_{2n}.$$

By the definition of the Bernoulli numbers and Theorem 2,

$$f(0) = f(1) = 0.$$

Suppose $f(x)$ changes sign in the open interval $(0, 1)$; then it must vanish for some x unequal to 0 or 1. By Rolle's Theorem $f'(x)$ vanishes for at least two points in the open interval $(0, 1)$. But

$$f'(x) = 2nB_{2n-1}(x),$$

and from Theorem 10, $B_{2n-1}(x)$ vanishes only at $x = 1/2$. This is a contradiction; therefore, $f(x)$ retains the same sign throughout the open interval $(0, 1)$.

POWER SERIES EXPANSION FOR THE TANGENT AND THE COTANGENT

In elementary calculus one learns how to find power series expansions for analytic functions through the use of Maclaurin's formula. For functions such as e^x, $\sin x$, $\cos x$, $(1+x)^{-1}$, and $\ln(1+x)$, the law of formation for each respective term is easily seen. When $\tan x$ is expanded by means of Maclaurin's formula, the law of formation of the individual terms is not readily seen. It will now be demonstrated that

the coefficients of the terms in the Maclaurin series expansion of tan x are essentially the Bernoulli numbers.

From the identity

$$y \coth y = iy \cot iy,$$

one has from equation (26)

$$iy \cot iy = \sum_{p=0}^{\infty} B_{2p} \frac{(2y)^{2p}}{(2p)!} \tag{34}$$

Setting x = iy and partially expanding the right-hand member of equation (34), one finds

$$x \cot x = B_0 + \sum_{p=1}^{\infty} B_{2p} \frac{(2x)^{2p}(-1)^p}{(2p)!} \tag{35}$$

Remembering that $B_0 = 1$, and dividing both sides of equation (35) by x, one has

$$\cot x = \frac{1}{x} + 2 \sum_{p=1}^{\infty} B_{2p} \frac{(2x)^{2p-1}(-1)^p}{(2p)!} \tag{36}$$

Using the identity

$$\tan x = \cot x - 2 \cot 2x$$

along with equation (36), one finds after some algebraic manipulation that

$$\tan x = 2 \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} (-1)^p \left[1 - 2^{2p}\right](2x)^{2p-1} \quad . \tag{37}$$

Now that power series expansions for cot x and tan x have been developed, the radii of convergence of these power series expansions will be examined. From Theorem 8,

$$B_{2p} = \frac{2(2p)!\,(-1)^{p-1}}{(2\pi)^{2p}} \zeta(2p) .$$

Since the test ratio will involve the limit of the quotient $\zeta(2p+2)/\zeta(2p)$, this limit will now be evaluated. Now,

$$\zeta(2p) = 1 + \frac{1}{2^{2p}} + \frac{1}{3^{2p}} + \cdots + \frac{1}{n^{2p}} + \cdots$$

and

$$1 + \left(\frac{1}{2^{2p}} + \frac{1}{3^{2p}}\right) + \left(\frac{1}{4^{2p}} + \frac{1}{5^{2p}} + \frac{1}{6^{2p}} + \frac{1}{7^{2p}}\right) + \cdots$$

$$< 1 + \left(\frac{1}{2^{2p}} + \frac{1}{2^{2p}}\right) + \left(\frac{1}{4^{2p}} + \frac{1}{4^{2p}} + \frac{1}{4^{2p}} + \frac{1}{4^{2p}}\right) + \cdots$$

$$= 1 + \frac{1}{2^{2p-1}} + \frac{1}{2^{4p-2}} + \cdots = \frac{1}{1 - 2^{-(2p-1)}} .$$

Therefore,

$$1 < \zeta(2p) < \frac{1}{1 - 2^{-(2p-1)}} .$$

Also

$$\zeta(2p+2) = 1 + \frac{1}{2^{2p+2}} + \frac{1}{3^{2p+2}} + \cdots + \frac{1}{n^{2p+2}}$$

and

$$1 + \left(\frac{1}{2^{2p+2}} + \frac{1}{3^{2p+2}}\right) + \cdots$$

$$< 1 + \left(\frac{1}{2^{2p}} + \frac{1}{2^{2p}}\right) + \left(\frac{1}{4^{2p}} + \frac{1}{4^{2p}} + \frac{1}{4^{2p}} + \frac{1}{4^{2p}}\right) + \cdots$$

$$= \frac{1}{1 - 2^{-(2p-1)}} \quad .$$

Thus,

$$1 < \zeta(2p+2) < \frac{1}{1 - 2^{-(2p-1)}} \quad .$$

Since

$$\lim_{p \to \infty} \frac{1}{1 - 2^{-(2p-1)}} = 1,$$

it follows that

$$\lim_{p \to \infty} \zeta(2p+2) = 1.$$

It can be shown in a similar manner that

$$\lim_{p \to \infty} \zeta(2p) = 1.$$

Since the limit of a quotient is the quotient of the limits, provided both limits exist,

$$\lim_{p \to \infty} \frac{\zeta(2p+2)}{\zeta(2p)} = 1.$$

Now, applying the ratio test to the power series expansion for $\cot x - \frac{1}{x}$, one has

$$L = \lim_{p\to\infty} \left| \frac{B_{2p+2}(2x)^{2p+1}(-1)^{p+1}}{(2p+2)!\ B_{2p}} \frac{(2p)!}{(2x)^{2p-1}(-1)^{p}} \right| .$$

$$= \lim_{p\to\infty} \left| \frac{2(2p+2)!\ (-1)^{p} \zeta(2p+2)(2x)^{2p+1}(-1)^{p+1}(2p)!(2\pi)^{2p}}{(2p+2)!\ (2\pi)^{2p+2} 2(2p)!\ (-1)^{p-1}(2x)^{2p-1}(-1)^{p}\zeta(2p)} \right|$$

$$= \lim_{p\to\infty} \frac{\zeta(2p+2)}{\zeta(2p)} \frac{x^2}{\pi^2} .$$

$$= \frac{x^2}{\pi^2} ,$$

because $\lim_{p\to\infty} \frac{\zeta(2p+2)}{\zeta(2p)} = 1.$

Thus the power series expansion for $\cot x - \frac{1}{x}$ converges for $x^2 < \pi^2$.

Similarly, it may be shown that the power series expansion for $\tan x$ converges for $x^2 < \frac{\pi^2}{4}$.

THE EULER-MACLAURIN FORMULA PRELIMINARY REMARKS

It is shown in the calculus that, if $f(x)$ is a function defined on the interval $[-a, a]$, and, if $f(x)$ has derivatives of all orders at $x = 0$, then $f(x)$ can be expanded in a Maclaurin series,

$$f(x) = f(0) + Df(0)x + D^2f(0)\frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n} D^n f(0)x^n. \tag{38}$$

If f(x) is analytic in the neighborhood of the origin, then the power series of equation (38) has a positive radius of convergence. Taylor's series with a remainder,

$$f(x) = f(0) + Df(0)x + D^2f(0)\frac{x^2}{2!} + \cdot \cdot \cdot + D^n f^n \frac{(0)x^n}{n!} + D^{n+1}\frac{f(t)\, x^{n+1}}{(n+1)!}$$

with $-a < t < a$ can be deduced on the assumption that f(x) has n+1 derivatives in $[-a, a]$. For the particular case where f(x) is a polynomial of degree less than or equal to n, equation (38) reduces to the finite series:

$$f(x) = f(0) + Df(0)x + D^2f(0)\frac{x^2}{2!} + \cdot \cdot \cdot + D^n f(0)\frac{x^n}{n!}\ . \tag{39}$$

In considering the problem of expanding a polynomial in terms of factorial polynomials one encounters Newton's formula[4],

$$f(x) = f(0) + \Delta f(0)x^{(1)} + \Delta^2 f(0)\frac{x^{(2)}}{2!} + \cdot \cdot \cdot + \Delta^n f(0)\frac{x^{(n)}}{n!}\ ,$$

which is analogous to equation (39).

The preceding discussion indicates that, given a function f(x) with certain properties, one can obtain an expansion of the function in terms of powers of x or factorial functions. There are many other types of expansions, such as a Fourier series, which express integrable periodic

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[4]Kenneth S. Miller, An Introduction to the Calculus of Finite Differences and Difference Equations, p. 18.

functions in terms of sines and cosines. In the theory of finite differences one might want to consider the possibility of expressing a certain function in terms of Bernoulli polynomials, i.e.,:

$$f(x) = a_0 B_0(x) + a_1 B_1(x) + a_2 B_2(x) + \cdots + a_n B_n(x) + \cdots. \tag{40}$$

The coefficients a_n can be determined by formal methods, as will be shown.

If one integrates both sides of equation (40) with respect to x over (0, 1), then

$$\int_0^1 B_n(x)dx = \frac{B_{n+1}(x)}{n+1} \Big|_0^1 = \frac{B_{n+1}(1)-B_{n+1}(0)}{n+1} = 0 \tag{41}$$

for $n = 1, 2, 3, \cdots$. Then one has

$$\int_0^1 f(x)dx = a_0 \int_0^1 B_0(x)dx = a_0 ,$$

since $B_0(x) = 1$. Thus a_0 has been determined. Now differentiate both sides of equation (40) to obtain

$$Df(x) = a_1 B_0(x) + 2a_2 B_1(x) + 3a_3 B_2(x) + \cdots + na_n B_{n-1}(x) + \cdots.$$

Integrating again from zero to one and using the results of equation (41), one has

$$\int_0^1 Df(x)dx = a_1 .$$

But

$$\int_0^1 Df(x)dx = f(x)\Big|_0^1 = f(1) - f(0) = \Delta f(0).$$

Therefore,

$$a_1 = \Delta f(0).$$

In general,

$$D^n f(x) = a_n n! B_0(x) + a_{n+1}(n+1)(n)(n-1)\cdot\cdot\cdot(2)B_1(x) + \cdot\cdot\cdot$$

and

$$\int_0^1 D^n f(x)dx = a_n n!\ .$$

Also

$$\int_0^1 D^n f(x)dx = D^{n-1}f(x)\Big|_0^1 = D^{n-1}\Delta f(0) = \Delta D^{n-1}f(0),$$

and

$$a_n = \frac{1}{n!}\ \Delta D^{n-1}f(0).$$

Thus, equation (40) may be written as

$$f(x) = \int_0^1 f(x)dx + \sum_{n=1}^{\infty} \frac{B_n(x)}{n!}\ \Delta D^{n-1}f(0). \tag{42}$$

If one makes the change of variable

$$t = x + y$$

and the change of function

$$f(x) = g(t),$$

then equation (42) becomes

$$g(x+y) = \int_{y}^{y+1} g(t)dt + \sum_{n=1}^{\infty} \frac{B_n(x)}{n!} \Delta D^{n-1} g(y).$$

If $x = 0$,

$$g(y) = \int_{y}^{y+1} g(t)\,dt + \sum_{n=1}^{\infty} \frac{B_n}{n!} \Delta D^{n-1}\, g(y), \tag{43}$$

and this is one form of the Euler-Maclaurin formula.

An alternate form of the Euler-Maclaurin formula that is useful in summation problems will now be developed. In order to do this, it is necessary to introduce a displacement operator E defined by the equation

$$Ef(x) = f(x + 1).$$

A relationship between the displacement operator E and the difference operator Δ must also be derived. By definition,

$$\begin{aligned}\Delta f(x) &= f(x+1) - f(x)\\ &= Ef(x) - f(x).\end{aligned}$$

Therefore,

$$(1 + \Delta)f(x) = Ef(x).$$

Suppose F(x) is a function such that

$$\Delta F(x) = f(x).$$

Then

$$F(x+1) - F(x) = f(x). \tag{44}$$

Summing equation (44) as x ranges from 0 to n-1 yields,

$$F(n) - F(0) = \sum_{x=0}^{n-1} f(x). \tag{45}$$

Expanding f(x+1) in a Taylor's series gives,

$$f(x+1) = \sum_{n=0}^{\infty} \frac{D^n f(x)}{n!} = \left[\sum_{n=0}^{\infty} \frac{D^n}{n!}\right] f(x) = e^D f(x).$$

As stated previously,

$$f(x+1) = Ef(x) = (1 + \Delta)f(x).$$

Hence, comparing the operators, one has

$$e^D = 1 + \Delta.$$

Introducing the operator Δ^{-1} one can rewrite

$$\Delta F(x) = f(x)$$

as

$$F(x) = \Delta^{-1} f(x) = \frac{1}{e^D - 1} f(x)$$

$$= \frac{D^{-1} Df(x)}{e^D - 1} \tag{46}$$

$$= D^{-1} \sum_{n=0}^{\infty} \frac{B_n D^n f(x)}{n!} .$$

Note that the last step follows from the definition of the Bernoulli numbers in equation (11). Expanding equation (46) and substituting in the numerical values of the Bernoulli numbers yields,

$$F(x) = \left[D^{-1} - \frac{1}{2} + \frac{D}{12} - \frac{D^3}{720} + \frac{D^5}{30240} - \cdots \right] f(x)$$

$$= D^{-1} f(x) - \frac{f(x)}{2} + \frac{f'(x)}{12} - \frac{f'''(x)}{720} + \cdots .$$

Evaluating this expression for x = 0 and x = n gives,

$$F(n) - F(0) = D^{-1}\left[f(n) - f(0)\right] - \frac{1}{2} f(n) + \frac{1}{2} f(0) + \frac{1}{12} f'(n) - \frac{1}{12} f'(0)$$

$$- \frac{1}{720} f'''(n) + \frac{1}{720} f'''(0) + \cdots . \tag{47}$$

The quantity $D^{-1}\left[f(n) - f(0)\right]$ can be replaced by $\int_0^n f(x)\, dx$, because

$$D^{-1}\left[f(n) - f(0)\right] = D^{-1} \int_0^n \frac{d}{dx} f(x) dx$$

$$= \int\int_0^n \frac{d}{dx} f(x) dx\, dx$$

$$= \int_0^n \int \frac{d}{dx} f(x)\, dx\, dx$$

$$= \int_0^n f(x)\, dx.$$

Recalling that

$$F(n) - F(0) = \sum_{x=0}^{n-1} f(x),$$

and adding f(n) to both sides of equation (47), one has,

$$\sum_{x=0}^{n} f(x) = \int_{0}^{n} f(x)\, dx + \frac{1}{2}\left[f(n) + f(0)\right] + \frac{1}{12}\left[f'(n) - f'(0)\right]$$

$$- \frac{1}{720}\left[f'''(n) - f'''(0)\right] + \cdots \; .$$

The Euler-Maclaurin formula can be expressed in several other forms that are also valuable in summation problems, and some of these other formulas can be written with a remainder term. The derivation of these additional formulas has been omitted as it is felt they are beyond the scope of this report.

CONCLUSION

The Bernoulli numbers and polynomials have many properties and uses other than those mentioned in this report. The Bernoulli numbers play an important role in the development of Stirling's formula which is used in statistics. Euler polynomials, which have extensive properties, can be defined in terms of Bernoulli polynomials.

It is possible to expand certain functions in Bernoulli series, and it is also possible to expand Bernoulli polynomials in a Fourier series over the interval (0, 1). As previously stated, the practical application of Bernoulli numbers and polynomials is in dealing with expansions and summation problems.

The discussion in the body of this report has been restricted to the derivation of properties of Bernoulli numbers and polynomials of the first kind. The reader should be aware that Bernoulli numbers and polynomials of the second kind exist, and they also possess generating functions and numerous properties.

REFERENCES

Fort, Tomlinson
Finite Differences and Difference Equations in the Real Domain. London: Oxford University Press, 1948.

Jordan, Charles.
Calculus of Finite Differences. New York: Chelsea Publishing Company, 1947.

Miller, Kenneth S.
An Introduction to the Calculus of Finite Differences and Difference Equations. New York: Henry Holt and Company, 1960.

Rudin, Walter.
Principles of Mathematical Analysis. London, New York, and Toronto: McGraw-Hill Book Company, Inc., 1953.

Stanton, Ralph G.
Numerical Methods for Science and Engineering. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961.

Taylor, Angus E.
Advanced Calculus, New York: Ginn and Company, 1955.

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BERNOULLI NUMBERS AND POLYNOMIALS

by

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AN ABSTRACT OF A MASTER'S REPORT

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This report is a brief study of the development of some of the major properties of Bernoulli numbers and polynomials of the first kind.

The Bernoulli numbers and polynomials both have generating functions, and both of these generating functions are defined. Several theorems concerning the more prominent properties of Bernoulli numbers and polynomials are proved. Maclaurin series expansions for the tangent and cotangent are derived illustrating the usefulness of Bernoulli numbers in these particular expansions. A relationship between the Bernoulli numbers and the Riemann-Zeta function is developed in closed form.

Two forms of the Euler-Maclaurin formula are deduced to demonstrate the value of Bernoulli numbers in their derivation. The report is concluded with a brief discussion regarding the use of Bernoulli numbers in other areas.